1. This exam contains 10 pages. Check that no pages are missing.

2. It is possible to collect up to 110 points. Try to collect as many points as possible.

3. Justify and prove all your answers (where applicable). Show all important steps in your solution.

4. All facts and results that were proved or stated in the class can be used in your solution without a proof. Such results need to be rigorously formulated.

5. Any printed and written material is allowed in the class. No electronic devices are allowed.

6. Exam duration is 2 hours.

7. Good luck!
Question 1 (20 points).

Take a language $\mathcal{L} \in \mathcal{P}$ defined over the alphabet $\Sigma = \{0, 1\}$. For each of the following languages determine if it is in $\mathcal{P}$ or not (justify your answers):

(a) $\mathcal{L}_1 = \left\{ w \underbrace{0 0 \cdots 0}_{25} : w \in \mathcal{L} \right\}$;

(b) $\mathcal{L}_2 = \left\{ \underbrace{ww \cdots w}_n : w \in \mathcal{L}, |w| = n \right\}$. 
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Question 2 (30 points).

Let $\Sigma = \{a,b,c\}$. Define the language $L_a$ over $\Sigma$:

$$L_a = \{ \langle A \rangle \mid A \text{ is a DFA, and the number of strings with } \text{“}a\text{” in } L(A) \text{ is } 17 \} .$$

Show that $L_a$ is a decidable language.
Question 3 (30 points).

Define a language \( L_\Sigma \) over the alphabet \( \Sigma = \{0, 1\} \):

\[
L_\Sigma = \{ \langle A \rangle \mid A \text{ is a DFA, and } L(A) = \Sigma^* \}.
\]

Show that \( L_\Sigma \in \mathcal{P} \).
Definition: consider an undirected finite graph $G(V, E)$. The graph $G$ is called a red-blue graph if every $v \in V$ has a color: either “red” or “blue”.

Define a language $(r, b)$-COLOR-CLIQUEs:

$$(r, b)$-COLOR-CLIQUEs = \left\{ \langle G, r, b \rangle \mid G \text{ is a red-blue graph} \right. $$

$$ $$

$$ \left. \text{ with a clique of size } r \text{ of red vertices and a clique of size } b \text{ of blue vertices} \right\}.

In this question, you will show that $(r, b)$-COLOR-CLIQUEs is $NP$-complete.

(a) Prove that $(r, b)$-COLOR-CLIQUEs $\in NP$.

(b) Prove that $(r, b)$-COLOR-CLIQUEs is $NP$-hard.

**Hint:** you can use a polynomial-time reduction from CLIQUE to $(r, b)$-COLOR-CLIQUEs. Do not forget to show that the reduction is correct and polynomial-time.