MTAT.03.260 Pattern Recognition and Image Analysis

Frequency Domain Filtering

Sven Laur
University of Tartu
Quick outline

Theoretical background
▷ Dirac delta function
▷ Continuous Fourier transformation
▷ Discrete Fourier transformation

Applications
▷ Reduction of aliasing and moire effects
▷ More informed design of low and highband filters
▷ Noise reduction by dedicated filters

Beyond frequency decomposition
▷ Image decomposition into localised basis function
▷ Image decomposition based on background knowledge
One-Dimensional Signal
and Its Spectrum
How to measure a signal?

All measurement devices average the intensity of a signal over some period. It can be modelled as convolution between a signal and a weight function.

\[ \hat{f}(t_0) = \int_{-\infty}^{\infty} f(t)w(t_0 - t)\,dt \]

where \( w(\cdot) \) is a weight function.

Most devices cannot see into the future: \( w(\tau) = 0 \) for \( \tau < 0 \) and forget quickly about the past: \( w(\tau) \ll 1 \) for \( \tau \gg 0 \).
An idealised measurement device

Dirac delta is the weight function of an idealised measurement device:

\[ \hat{f}(t_0) = \int_{-\infty}^{\infty} f(t) \delta(t_0 - t) \, dt = f(t_0). \]

This device cannot be built nor does the Dirac delta function \( \delta \) exist in a standard sense. Hence, we have to model it as a limiting process.

\[ \delta_i(t) = \begin{cases} i & \text{if } -\frac{1}{i} \leq t \leq \frac{1}{i} , \\ 0, & \text{otherwise} . \end{cases} \]
How to represent digitalised signal?

A digital signal is often represented as a step function. However, this is not accurate, since we do not know nothing beyond sampled values.

\[ f_s(t) = \sum_{i=-\infty}^{\infty} f(t_i) \delta(t - t_i) = f(t) \cdot \sum_{i=-\infty}^{\infty} \delta(t - t_i) \]

This representation makes it formally correct to specify the convolution of original and sampled signal using continuous integrals.
A convenient way to represent sine waves

A sine wave is determined by its amplitude $a$, frequency $\omega$ and phase $\phi$

$$f_{\text{sine}}(t) = a \cdot \sin(\omega x + \phi) .$$

Due to Euler’s formula $\exp(jx) = \cos(x) + j \sin(x)$ we can represent sine waves as real or complex parts of an exponent function.

$$f(t) = \Im(a \cdot \exp(\phi + j\omega t))$$
$$f(t) = \Re(a \cdot \exp(\phi + j\omega t - \pi/2))$$

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Can any signal be split into sine waves?

**Periodic functions.** All *well-behaving* functions with a period $T$ can be represented as an infinite series of sine waves:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp\left(j \cdot \frac{2\pi n}{T} \cdot t\right)$$

where

$$c_n = \frac{1}{T} \cdot \int_{-T/2}^{T/2} f(t) \exp\left(-j \cdot \frac{2\pi n}{T} \cdot t\right) \, dt.$$  

It is sufficient that $f$ has left and right derivatives in each point.
An example of a spectral decomposition

\[ F(\mu) = \sum_{n=-\infty}^{\infty} c_n \cdot \delta(\mu - n) \]

\[ = - \sum_{n=-\infty}^{\infty} \frac{\pi j}{2n} \cdot \delta(\mu - n) \]
What if the function is not periodic?

Linear combination of multiples of base frequencies always produces a periodic function. Hence, arbitrary function must have continuous spectrum.

For well-behaving functions, the spectral decomposition can be computed with the Fourier transform

\[ F(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \exp(-j2\pi\mu t) dt . \]

The inverse Fourier transform converts a spectrum back to the signal

\[ f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\mu) \exp(j2\pi\mu t) d\mu . \]

For the existence, it is sufficient that \[ \int_{-\infty}^{\infty} |f(t)| dt < \infty. \]
Examples of spectral decomposition
Examples of spectral decomposition

\[ f(t) = \sum_{n} \delta(t - n) \]

\[ F(\mu) = \sum_{n} \delta(\mu - n) \]
Basic properties of Fourier transform

Let $\mathcal{F}(\cdot)$ denote the Fourier transform in the following statements.

▷ Fourier transform is linear, i.e., for any $\alpha, \beta \in \mathbb{C}$

$$\mathcal{F}(\alpha f + \beta g) = \alpha \mathcal{F}(f) + \beta \mathcal{F}(g) .$$

▷ Fourier transform groups functions into pairs

$$g(\mu) = \mathcal{F}(f(t)) \iff f(-t) = \mathcal{F}^{-1}(g(\mu)) .$$

▷ Fourier transform reduces convolution into a product and vice versa:

$$\mathcal{F}(f \ast g) = \mathcal{F}(f) \cdot \mathcal{F}(g) \quad \text{and} \quad \mathcal{F}(f \cdot g) = \mathcal{F}(f) \ast \mathcal{F}(g) .$$

▷ Scaling, phase-shifts and translations have simple formulae.
What if the signal is digital?

As the signal is digital, the Fourier transform reduces into a sum

\[ F(\mu) = \sum_{n=\infty}^{\infty} f(n\Delta t) \exp(-j2\pi \mu n \Delta T') . \]

The function is periodic and allows to reconstruct all values.

The entire spectrum is not needed for the reduction of the first \( M \) values \( f(0), \ldots, f(M - 1) \). It is sufficient to sample \( F(\mu) \) in the grid

\[
\begin{array}{cccc}
0 & 1 & \cdots & M - 1 \\
M\Delta T' & M\Delta T' & \cdots & M\Delta T' \\
\end{array}
\]

The resulting transform is known as *discrete Fourier transform*. 
Geometrical illustration

Formally, the discrete Fourier transformation is defined as follows

\[ F_m = \sum_{n=0}^{M-1} f_n \exp\left(-j\frac{2\pi n}{M} \cdot m\right) \iff f_n = \frac{1}{M} \cdot \sum_{m=0}^{M-1} f_m \exp\left(j\frac{2\pi m}{M} \cdot n\right). \]

As a result, the decomposition task is stated in a standardised grid.
What signal components are inherently lost?

As signal and its spectrum different representations of the same object, we can compare the information content of the spectra.

At sampling rate $\Delta T$ the spectrum of discretised signal can be represented as a convolution

$$F_s(\mu) = F(\mu) \ast \frac{1}{\Delta T} \cdot \sum_{n=-\infty}^{\infty} \delta(\mu - \frac{n}{\Delta T}) = \frac{1}{\Delta T} \cdot \sum_{n=\infty}^{\infty} F(\mu - \frac{n}{\Delta T}).$$

Hence if the original spectra is mostly contained in the region

$$\left[-\frac{1}{\Delta T}, \frac{1}{\Delta T}\right]$$

then most of the signal can be reliably reconstructed.
Two-dimensional Fourier Transform
Continuous Fourier transform

The Fourier transform of a two-dimensional signal \( f(t, z) \) is defined

\[
F(\mu, \nu) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t, z) \exp(-j2\pi(\mu t + \nu z)) dt \, dz
\]

and the inverse Fourier transform is defined

\[
f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu) \exp(j2\pi(\mu t + \nu z)) d\mu \, d\nu.
\]

The transformations have analogous properties to the one-dimensional counterparts. In particular, it is linear and splits convolution into a product.
Aliasing problems

Let $\Delta T$ and $\Delta Z$ be sampling intervals. Then whenever a the relevant part of a spectrum of a sampled image does not fit into the frequency square

$$\left[ -\frac{1}{2\Delta T}, \frac{1}{2\Delta T} \right] \times \left[ -\frac{1}{2\Delta Z}, \frac{1}{2\Delta Z} \right]$$

various can occur in the image.

▷ The effect manifest itself in scanning of real images.
▷ The effect manifest itself in resampling of digital images.
Discrete Fourier transform

The discrete Fourier transform of a two-dimensional signal \( f(t, z) \) is defined

\[
F(u, v) = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} f(x, y) \exp(-j2\pi(ux/M + vy/N))
\]

and the inverse Fourier transform is defined

\[
f(x, y) = \frac{1}{MN} \cdot \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} F(u, v) \exp(j2\pi(ux/M + vy/N))
\]

The transformations have analogous properties to the one-dimensional counterparts. In particular, it is linear and splits convolution into a product.
Properties of Fourier spectrum

As the Fourier spectrum is complex-valued function, we can represent the spectrum in terms of magnitude and phase angle

\[ M(u, v) = |F(u, v)| \quad \phi(u, v) = \arctan\left(\frac{\Im(F(u, v))}{\Re(F(u, v))}\right) \]

The power spectrum is defined as the square of magnitude

\[ P(u, v) = |F(u, v)|^2 \]

Phase angle encodes most relevant features of an image.

- Phase distortion for low frequencies causes significant artefacts.
- Magnitude distortion causes much milder noise patterns.
Wrap-around errors

When we perform the discrete Fourier transform to compute convolution as multiplication, the periodicity introduces wrap-around errors.

\[ f(x, y) \ast g(x, y) = \sum_{m=0}^{M} \sum_{n=0}^{N} f(x, y)g(x - m \mod M, y - n \mod N) \]

To get rid of these terms, we have to pad functions with zeroes.
Frequency Domain
Filters
General philosophy

- Disturbances in phases cause significant changes in images.
- Changes in modules provide more predictable results.
- It reasonable to use zero-phase-shift filters.
- Multiplicative filters in frequency domain can be restated as convolution filters in spatial domain and vice versa.
- The connection can be used for analysing filter properties.
Low-pass filters

▷ Ideal low-pass filter

\[ H(u, v) = \begin{cases} 1, & \text{if } u^2 + v^2 \leq D_0^2, \\ 0, & \text{otherwise}. \end{cases} \]

▷ Butterworth filter

\[ H(u, v) = \frac{1}{1 + \left(\frac{u^2 + v^2}{D_0^2}\right)^n}. \]

▷ Gaussian filter

\[ H(u, v) = \exp\left(-\frac{u^2 + v^2}{2D_0^2}\right). \]
Aftermath

▷ Ideal low-pass filter creates ringing lines near the borders.
▷ Butteworth filter is immune against ringing for $n = 1, 2$.
▷ Gaussian filter does not introduce ringing.
▷ A good filter is local in the spatial domain.
High-pass filters

▷ Ideal high-pass filter

\[
H(u, v) = \begin{cases} 
0, & \text{if } u^2 + v^2 \leq D_0^2, \\
1, & \text{otherwise}. 
\end{cases}
\]

▷ Butterworth filter

\[
H(u, v) = 1 - \frac{1}{1 + \left(\frac{u^2 + v^2}{D_0^2}\right)^n}.
\]

▷ Gaussian filter

\[
H(u, v) = 1 - \exp\left(-\frac{u^2 + v^2}{2D_0^2}\right).
\]
Special filters for repeating patterns

- Sometimes the noise has a distinct spectral pattern.
- Then we can design a special filter that reduces these frequencies.
- If the original image does not span to these areas of spectrum, then noise reduction is rather good.
- Method is guaranteed to work only for additive noise.
Multiplicatively homomorphic filters

- Sometimes the noise is multiplicative.
- Then all previous methods provide suboptimal results.
- Logarithmic transform makes the noise additive again.
- All previous tricks become applicable again.