Advanced Algorithmics (4AP)  
Heaps  

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2009 Spring
Heaps


• Binary heap

• Binomial heap

• Fibonacci heap
Heap/Priority queue

• Find min/Delete; Insert;

• Decrease key (change value of the key)

• Merge two heaps ...
As the table in Figure 19.1 shows, if we don't need the UNION operation, ordinary binary heaps, as used in heapsort (Chapter 6), work well. Operations other than UNION run in worst-case time $O(\lg n)$ (or better) on a binary heap. If the UNION operation must be supported, however, binary heaps perform poorly. By concatenating the two arrays that hold the binary heaps to be merged and then running MIN-HEAPIFY (see Exercise 6.2-2), the UNION operation takes $T(n)$ time in the worst case.

<table>
<thead>
<tr>
<th>Procedure</th>
<th>Binary heap (worst-case)</th>
<th>Binomial heap (worst-case)</th>
<th>Fibonacci heap (amortized)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAKE-HEAP</td>
<td>$T(1)$</td>
<td>$T(1)$</td>
<td>$T(1)$</td>
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<tr>
<td>INSERT</td>
<td>$T(\lg n)$</td>
<td>$O(\lg n)$</td>
<td>$T(1)$</td>
</tr>
<tr>
<td>MINIMUM</td>
<td>$T(1)$</td>
<td>$O(\lg n)$</td>
<td>$T(1)$</td>
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<td>EXTRACT-MIN</td>
<td>$T(\lg n)$</td>
<td>$T(\lg n)$</td>
<td>$O(\lg n)$</td>
</tr>
<tr>
<td>UNION</td>
<td>$T(n)$</td>
<td>$O(\lg n)$</td>
<td>$T(1)$</td>
</tr>
<tr>
<td>DECREASE-KEY</td>
<td>$T(\lg n)$</td>
<td>$T(\lg n)$</td>
<td>$T(1)$</td>
</tr>
<tr>
<td>DELETE</td>
<td>$T(\lg n)$</td>
<td>$T(\lg n)$</td>
<td>$O(\lg n)$</td>
</tr>
</tbody>
</table>

**Figure 19.1**: Running times for operations on three implementations of mergeable heaps. The number of items in the heap(s) at the time of an operation is denoted by $n$. 


Binomial heaps:

• **Performance:** All of the following operations work in $O(\log n)$ time on a binomial heap with $n$ elements:
  - Insert a new element to the heap
  - Find the element with minimum key
  - Delete the element with minimum key from the heap
  - Decrease key of a given element
  - Delete given element from the heap
  - Merge two given heaps to one heap
  - Finding the element with minimum key can also be done in $O(1)$ by using an additional pointer to the minimum.
• http://www.cse.yorku.ca/~aaw/Jason/FibonacciHeapAnimation.html


• http://www.jucs.org/jucs_7_5/animation_for_teaching_purposes/Lauer_T.html
Binomial heaps, Fibonacci heaps, and applications

Dan Feldman
• CLRS:
Binomial trees

$B_0$

$B_1$

$B_i$

$B_{(i-1)}$

$B_{(i-1)}$
Binomial trees
19.1.1 Binomial trees

The binomial tree \( B_k \) is an ordered tree (see Section B.5.2) defined recursively. As shown in Figure 19.2(a), the binomial tree \( B_0 \) consists of a single node. The binomial tree \( B_k \) consists of two binomial trees \( B_{k-1} \) that are linked together: the root of one is the leftmost child of the root of the other. Figure 19.2(b) shows the binomial trees \( B_0 \) through \( B_4 \).

Figure 19.2: (a) The recursive definition of the binomial tree \( B_k \). Triangles represent rooted subtrees. (b) The binomial trees \( B_0 \) through \( B_4 \). Node depths in \( B_4 \) are shown. (c) Another way of looking at the binomial tree \( B_k \).
Lemma 20.1

• For the binomial tree $B_k$,
• 1. there are $2^k$ nodes,
• 2. the height of the tree is $k$,
• 3. there are exactly $\binom{k}{i}$ nodes at depth $i$ for $i = 0, 1, \ldots, k$, and
• 4. the root has degree $k$, which is greater than that of any other node; moreover if the children of the root are numbered from left to right by $k - 1, k - 2, \ldots, 0$, child $i$ is the root of a subtree $B_i$. 
Properties of binomial trees

1) \( |B_k| = 2^k \)
2) \( \text{degree}(\text{root}(B_k)) = k \)
3) \( \text{depth}(B_k) = k \)

\[ \Rightarrow \] The degree and depth of a binomial tree with at most \( n \) nodes is at most \( \log(n) \).

Define the rank of \( B_k \) to be \( k \)
Figure 20.4 The binomial tree $B_4$ with nodes labeled in binary by a postorder walk.
Binomial heaps (def)

A collection of binomial trees at most one of every rank.
Items at the nodes, heap ordered.

Possible rep: Doubly link roots and children of every node. Parent pointers needed for delete.
Figure 19.3: A binomial heap $H$ with $n = 13$ nodes. (a) The heap consists of binomial trees $B_0, B_2,$ and $B_3$, which have 1, 4, and 8 nodes respectively, totaling $n = 13$ nodes. Since each binomial tree is min-heap-ordered, the key of any node is no less than the key of its parent. Also shown is the root list, which is a linked list of roots in order of increasing degree. (b) A more detailed representation of binomial heap $H$. Each binomial tree is stored in the left-child, right-sibling representation, and each node stores its degree.
Binomial heaps (operations)

Operations are defined via a basic operation, called **linking**, of binomial trees:
Produce a $B_k$ from two $B_{k-1}$, keep heap order.
Binomial heaps (ops cont.)

Basic operation is **meld(h1,h2):**

Like addition of binary numbers.

\[
\begin{align*}
\text{h1:} & \quad B_5 \quad B_4 \quad B_3 \quad B_2 \quad B_1 \\
\text{h2:} & \quad B_4 \quad B_3 \quad \quad \quad B_0 \\
\text{meld(h1,h2):} & \quad B_5 \quad B_4 \quad B_2 \quad B_0 \\
\end{align*}
\]
The execution of BINOMIAL-HEAP-UNION.(a) Binomial heaps $H_1$ and $H_2$. 

(a) $\text{head}[H_1] \rightarrow 12 \rightarrow 7 \rightarrow 15$  
$\qquad \qquad \qquad \quad \rightarrow 28 \rightarrow 23 \rightarrow 41$ 
$\text{BINOMIAL-HEAP-MERGE}$ 

(b) $\text{head}[H] \rightarrow 12^x \rightarrow 18 \rightarrow 7 \rightarrow 3 \rightarrow 15$ 
$\qquad \qquad \quad \rightarrow 28 \rightarrow 33 \rightarrow 41$ 
$\text{Case 3}$ 

(c) $\text{head}[H] \rightarrow 12 \rightarrow 7 \rightarrow 3 \rightarrow 15$ 
$\qquad \qquad \qquad \quad \rightarrow 28 \rightarrow 33 \rightarrow 41$ 
$\text{Case 2}$
The execution of BINOMIAL-HEAP-UNION.(a) Binomial heaps $H_1$ and $H_2$. (continued)
Delete min

Find min (=1)

Extract tree

Split tree, reverse

Merge/meld
Decrease key (y=26 => y=7)
Binomial heaps (ops cont.)

Findmin(h): obvious

Insert(x,h) : meld a new heap with a single B₀ containing x, with h

deletemin(h) : Chop off the minimal root. Meld the subtrees with h. Update minimum pointer if needed.

delete(x,h) : Bubble up and continue like delete-min

decrease-key(x,h,δ) : Bubble up, update min ptr if needed

All operations take O(log n) time on the worst case, except find-min(h) that takes O(1) time.
Amortized analysis

We are interested in the worst case running time of a sequence of operations.

Example: binary counter

single operation -- increment

00000
00001
00010
00011
00100
00101
Amortized analysis (Cont.)

On the worst case increment takes $O(k)$.

$k = \#\text{digits}$

What is the complexity of a sequence of increments (on the worst case)?

Define a potential of the counter:

$$\Phi (c) = ?$$

Amortized(increment) = actual(increment) + $\Delta \Phi$
Amortized analysis (Cont.)

Amortized(increment$_1$) = actual(increment$_1$) + $\Phi_1 - \Phi_0$
Amortized(increment$_2$) = actual(increment$_2$) + $\Phi_2 - \Phi_1$
    $$\vdots$$
    $$\vdots$$
Amortized(increment$_n$) = actual(increment$_n$) + $\Phi_n - \Phi_{(n-1)}$

$$\sum_i Amortized(increment_i) = \sum_i actual(increment_i) + \Phi_n - \Phi_0$$
$$\sum_i Amortized(increment_i) \geq \sum_i actual(increment_i)$$
if $\Phi_n - \Phi_0 \geq 0$
Amortized analysis (Cont.)

Define a potential of the counter:

\[ \Phi(c) = \#(\text{ones}) \]

Amortized(increment) = actual(increment) + \( \Delta \Phi \)

Amortized(increment) = 1 + \#(1 \Rightarrow 0) + 1 - \#(1 \Rightarrow 0) = O(1)

===> Sequence of \( n \) increments takes \( O(n) \) time
Binomial heaps - amortized ana.

\[ \Phi (\text{collection of heaps}) = \#(\text{trees}) \]

Amortized cost of insert \( O(1) \)
Amortized cost of other operations still \( O(\log n) \)
Allow more than one tree of each rank.

Meld \((h_1, h_2)\) :

- Concatenate the lists of binomial trees.
- Update the minimum pointer to be the smaller of the minimums

\(O(1)\) worst case and amortized.
Binomial heaps + lazy meld

As long as we do not do a delete-min our heaps are just doubly linked lists:

```
9  5  9  11  4  6
```

Delete-min: Chop off the minimum root, add its children to the list of trees.

**Successive linking**: Traverse the forest keep linking trees of the same rank, maintain a pointer to the minimum root.
Binomial heaps + lazy meld

Possible implementation of delete-min is using an array indexed by rank to keep at most one binomial tree of each rank that we already traversed.

Once we encounter a second tree of some rank we link them and keep linking until we do not have two trees of the same rank. We record the resulting tree in the array

\[
\text{Amortized}(\text{delete-min}) =
\]

\[
= (\text{#links} + \text{max-rank}) - \text{#links}
\]

\[
= O(\log(n))
\]
Fibonacci heaps (Fredman & Tarjan 84)

Want to do decrease-key(x,h,δ) faster than delete+insert. Ideally in O(1) time.

Why?
Dijkstra’s shortest path algorithm

Let $G = (V,E)$ be a weighted (weights are non-negative) undirected graph, let $s \in V$. Want to find the distance (length of the shortest path), $d(s,v)$ from $s$ to every other vertex.

![Graph diagram with weights labeled at each edge]
Figure 20.1: (a) A Fibonacci heap consisting of five min-heap-ordered trees and 14 nodes. The dashed line indicates the root list. The minimum node of the heap is the node containing the key 3. The three marked nodes are blackened. The potential of this particular Fibonacci heap is 5+2·3 = 11. (b) A more complete representation showing pointers $p$ (up arrows), $child$ (down arrows), and $left$ and $right$ (sideways arrows). These details are omitted in the remaining figures in this chapter, since all the information shown here can be determined from what appears in part (a).
Insert (left from root)
Finding the minimum node

• The minimum node of a Fibonacci heap $H$ is given by the pointer $min[H]$, so we can find the minimum node in $O(1)$ actual time. Because the potential of $H$ does not change, the amortized cost of this operation is equal to its $O(1)$ actual cost.
Figure 21.3 The action of FIB-HEAP-EXTRACT-MIN.

A Fibonacci heap H. (b) The situation after the minimum node z is removed from the root list and its children are added to the root list. (c)-(e) The array A and the trees after each of the first three iterations of the for loop of lines 3-13 of the procedure CONSOLIDATE. The root list is processed by starting at the minimum node and following right pointers. Each part shows the values of w and x at the end of an iteration. (f)-(h) The next iteration of the for loop, with the values of w and x shown at the end of each iteration of the while loop of lines 6-12. Part (f) shows the situation after the first time through the while loop. The node with key 23 has been linked to the node with key 7, which is now pointed to by x. In part (g), the node with key 17 has been linked to the node with key 7, which is still pointed to by x. In part (h), the node with key 24 has been linked to the node with key 7. Since no node was previously pointed to by A[3], at the end of the for loop iteration, A[3] is set to point to the root of the resulting tree. (i)-(l) The situation after each of the next four iterations of the while loop. (m) Fibonacci heap H after reconstruction of the root list from the array A and determination of the new min[H] pointer.
(a) The initial Fibonacci heap. (b) The node with key 46 has its key decreased to 15. The node becomes a root, and its parent (with key 24), which had previously been unmarked, becomes marked. (c)-(e) The node with key 35 has its key decreased to 5. In part (c), the node, now with key 5, becomes a root. Its parent, with key 26, is marked, so a cascading cut occurs. The node with key 26 is cut from its parent and made an unmarked root in (d). Another cascading cut occurs, since the node with key 24 is marked as well. This node is cut from its parent and made an unmarked root in part (e). The cascading cuts stop at this point, since the node with key 7 is a root. (Even if this node were not a root, the cascading cuts would stop, since it is unmarked.) The result of the FIB-HEAP-DECREASE-KEY operation is shown in part (e), with min[H] pointing to the new minimum node.
Application #2 : Prim’s algorithm for MST

Start with T a singleton vertex.
Grow a tree by repeating the following step:
Add the minimum cost edge connecting a vertex in T to a vertex out of T.
Application #2 : Prim’s algorithm for MST

Maintain the vertices out of T but adjacent to T in a heap.

The key of a vertex \( v \) is the weight of the lightest edge \((v,w)\) where \( w \) is in the tree.

Iteration: Do a delete-min. Let \( v \) be the minimum vertex and \((v,w)\) the lightest edge as above. Add \((v,w)\) to T. For each edge \((w,u)\) where \( u \notin T\),

if \( \text{key}(u) = \infty \) insert \( u \) into the heap with \( \text{key}(u) = w(w,u) \)
if \( w(w,u) < \text{key}(u) \) decrease the key of \( u \) to be \( w(w,u) \).

With regular heaps \( O(m \log(n)) \).
With F-heaps \( O(n \log(n) + m) \).
Suggested implementation for decrease-key(x,h,δ):

If x with its new key is smaller than its parent, **cut the subtree rooted at x and add it to the forest**. Update the minimum pointer if necessary.
Decrease-key (cont.)

Does it work?

Obs1: Trees need not be binomial trees any more.

Do we need the trees to be binomial?
Where have we used it?

In the analysis of delete-min we used the fact that at most \(\log(n)\) new trees are added to the forest. This was obvious since trees were binomial and contained at most \(n\) nodes.
Decrease-key (cont.)

Such trees are now legitimate. So our analysis breaks down.
Fibonacci heaps (cont.)

We shall allow non-binomial trees, but will keep the degrees logarithmic in the number of nodes.

Rank of a tree = degree of the root.

Delete-min: do successive linking of trees of the same rank and update the minimum pointer as before.

Insert and meld also work as before.
Fibonacci heaps (cont.)

Decrease-key \((x,h,\delta)\): indeed cuts the subtree rooted by \(x\) if necessary as we showed.

in addition we maintain a mark bit for every node. When we cut the subtree rooted by \(x\) we check the mark bit of \(p(x)\). If it is set then we cut \(p(x)\) too. We continue this way until either we reach an unmarked node in which case we mark it, or we reach the root.

This mechanism is called cascading cuts.
Fibonacci heaps (delete)

Delete(x, h) : Cut the subtree rooted at x and then proceed with cascading cuts as for decrease key.

Chop off x from being the root of its subtree and add the subtrees rooted by its children to the forest.

If x is the minimum node do successive linking.
• In the **amortized running time** analysis we pretend that very fast operations take a little bit longer than they actually do.

• This additional time is then later subtracted from the actual running time of slow operations.

• The amount of time saved for later use is measured at any given moment by a potential function.
• **The potential of a Fibonacci heap** is given by

• Potential = $t + 2m$ where $t$ is the number of trees in the Fibonacci heap, and $m$ is the number of marked nodes. A node is marked if at least one of its children was cut since this node was made a child of another node (all roots are unmarked).
Fibonacci heaps (analysis)

Want everything to be $O(1)$ time except for delete and delete-min.

$\implies$ cascading cuts should pay for themselves

$\Phi$ (collection of heaps) = #$\text{trees}) + 2#$ (marked nodes)

Actual$(\text{decrease-key}) = O(1) +$ # (cascading cuts)
$\Delta \Phi$(decrease-key) = $O(1)$ - # (cascading cuts)

$\implies$ amortized$(\text{decrease-key}) = O(1)$!
Fibonacci heaps (analysis)

What about delete and delete-min?

Cascading cuts and successive linking will pay for themselves. The only question is what is the maximum degree of a node? How many trees are being added into the forest when we chop off a root?
Fibonacci heaps (analysis)

Lemma 1: Let x be any node in an F-heap. Arrange the children of x in the order they were linked to x, from earliest to latest. Then the i-th child of x has rank at least i-2.

Proof:
When the i-th node was linked it must have had at least i-1 children.
Since then it could have lost at most one.
Fibonacci heaps (analysis)

Corollary 1: A node $x$ of rank $k$ in a F-heap has at least $\phi^k$ descendants, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio.

Proof:

Let $s_k$ be the minimum number of descendants of a node of rank $k$ in a F-heap.

By Lemma 1 $s_k \geq \sum_{i=0}^{k-1} s_i + 2$

$s_0 = 1, s_1 = 2$
Figure 20.1: (a) A Fibonacci heap consisting of five min-heap-ordered trees and 14 nodes. The dashed line indicates the root list. The minimum node of the heap is the node containing the key 3. The three marked nodes are blackened. The potential of this particular Fibonacci heap is $5 + 2 \cdot 3 = 11$. (b) A more complete representation showing pointers $p$ (up arrows), child (down arrows), and left and right (sideways arrows). These details are omitted in the remaining figures in this chapter, since all the information shown here can be determined from what appears in part (a).
Fibonacci heaps (analysis)

Proof (cont):

Fibonacci numbers satisfy

\[ F_{k+2} = \sum_{i=2}^{k} F_i + 2, \text{ for } k \geq 2, \text{ and } F_2 = 1 \]

so by induction \( s_k \geq F_{k+2} \)

It is well known that \( F_{k+2} \geq \phi^k \)

It follows that the maximum degree \( k \) in a F-heap with \( n \) nodes is such that \( \phi^k \leq n \)

so \( k \leq \log(n) / \log(\phi) = 1.4404 \log(n) \)
Fibonacci heaps and other amortized data structures may not be appropriate for real-time systems.

### Summary of running times

<table>
<thead>
<tr>
<th></th>
<th>Linked List</th>
<th>Binary Tree</th>
<th>(Min-)Heap</th>
<th>Fibonacci Heap</th>
<th>Brodal Queue [1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>insert</td>
<td>O(1)</td>
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<td>O(log n)</td>
<td>O(1)*</td>
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<tr>
<td>delete</td>
<td>O(n)</td>
<td>O(n)</td>
<td>O(log n)</td>
<td>O(log n)*</td>
<td>O(log n)</td>
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<td>merge</td>
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<td>O(m log(n+m))</td>
<td>O(m log(n+m))</td>
<td>O(1)</td>
<td>O(1)</td>
</tr>
</tbody>
</table>

(*)Amortized time
• C code
  – http://www.cs.unc.edu/~bbb/foss/binheaps/iheap.h
  – http://www.cs.unc.edu/~bbb/#binomial_heaps
More...

- [http://www.cse.yorku.ca/~aaw/Jason/FibonacciHeapAnimation.html](http://www.cse.yorku.ca/~aaw/Jason/FibonacciHeapAnimation.html)